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THE HISTORY OF MATHEMATICS*

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THE earliest dawn of science is without doubt not different from that of intelligence. But the civilized man of to-day, far removed as he is from the lowest of existing human races, is probably as far again from the being whom one would not differentiate from the animals as far as mental powers are concerned. What this difference is, neither ethnologist nor psychologist can yet tell. Perhaps the nearest approach to a definition, at least from the point of view of this article, is contained in the distinction between unconscious and conscious observation. We are familiar with both sides even in ourselves: records can be impressed on the brain and remain there apparently dormant until some stimulus brings them to fruition, and again, record and stimulus can appear together so that a train of thought is immediately started.

The faculty of conscious observation is a fundamental requirement of a scientific training, and no development can take place until it has been acquired to some extent. Although this is only the first step, it was probably a long one in the history of the race, just as it is relatively long in the lives of the majority of individuals. Reasoning concerning the observation follows, but not much success can be attained until a considerable number of observations have been accumulated. We may indeed put two and two together to make four, but experience shows that with phenomena the answer is more often wrong than right: we need much more information in order to get a correct answer.

We expect, then, that the earlier stage of a science will be one not so much of discovery as of conscious observation of phenomena which are apparent as soon as attention is called to them. The habit once acquired, the search for the less obvious facts of nature begins, and

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in the search many unexpected secrets are found. Once a body of facts has been accumulated, correlation follows. The attempt is made to find something common to them all and it is at this stage that science, in the modern sense, may perhaps be said to have its birth. But this is only a beginning. The mind that can grasp correlation can soon proceed to go further and try to find a formula which will not only be a common property, but which will completely embrace the facts; that is, in modern parlance, a law which groups all the phenomena under one head.

The formula or law once discovered, consequences other than those known are sought, and the process of scientific discovery begins. Realms which could never have been opened up by observation alone are revealed to the mind which has the ability to predict results as consequences of the law, and thence is found the means by which the truth of the law is tested. If the further consequences are shown to be in agreement with what may be observed, the evidence is favorable. If the contrary, the law must be abandoned or changed so as to embrace the newly discovered phenomena. The process of trial and error, or of hypothesis and test, is a recurring one which embraces a large proportion of the scientific work of to-day.

Scientific development has two main aspects. One is the framing of laws in order to discover new phenomena and develop the subject forward so as to open out new roads into the vast forest of the secrets of nature. The other is the turning backward in order to discover the foundations on which the science rests. Just as no teacher would think it wise to start the young pupil in chemistry or physics by introducing him at the outset to the fundamental unit of matter or energy as it is known at the time, but will rather start in the middle of the subject with facts which are within the comprehension of his mind and the experience of his observation, science itself has been and must necessarily be developed in the same manner. We proceed down to the foundations as well as up to the phenomena.

This two-fold aspect of scientific research has had revolutionary results in the experience of the last half century. It has fundamentally changed the ideas of those who study the so-called laboratory subjects in which observation with artificially constructed materials goes hand in hand with the framing of laws and hypotheses, but it has changed the study of mathematics in an even more fundamental manner. In the past, geometry and arithmetic were suggested by observation and practical needs and the development of both with symbolic representation proceeded on lines which were dictated by the problems which arose. The methods of discovery in working forward were not essentially different from those of an observational science, except perhaps that the testing of a new law was unnecessary on account of the

rules of reasoning which accompanied them and became embodied into a system of logic. It was, however, in proceeding backward to discover the foundations that the whole aspect of the subject changed. While many of the hypotheses were suggested by observation and were known by numerous tests to be applicable to the discussion of natural phenomena, it became evident that the actual hypotheses used were independent of the phenomena. The laws which are at the basis of geometrical reasoning are not necessarily natural laws: they can equally well be regarded as mere productions of the brain in the same sense as we might imagine a race of intelligent beings on another planet free from some of our limitations or restricted by limitations from which we are free. It is seen to be the same with the rules of symbolic reasoning which have gradually grown up. A geometry without Euclid's axiom of parallels has been constructed perfectly consistent in all its parts. This is built up of a set of axioms which constitute its foundation together with a code of reasoning by which we develop the consequences of the axioms. The same is true of geometries which involve space in more than three dimensions. It is somewhat easier to imagine symbolic developments which have their foundations different from those of our school and college algebras because there is no obvious connection between these rules and the phenomena of nature.

It may be asked what limitation is there in the development of mathematical theories if any set of axioms may be laid down. Theoretically there is none, except that if we retain our code of reasoning about them such axioms must not be inconsistent with one another. A certain sense of the fitness of things restrains mathematicians from a wild overturning of the law and order which have been established in the development of mathematics, just as it restrains democracies from trying experiments in government which overturn too much the existing order of affairs. Changes proceed in mathematics just as in politics by evolution rather than by revolution. The slowly built up structure of the past is not to be lightly overturned for the sake of novelty.

The developments traced above apply mainly to the subject of mathematics apart from its applications to the solution of problems presented by nature. Applied mathematics is a method of reasoning through symbols by which we can discover the consequences of the assumed laws of nature. The symbolism which we adopt and the rules we lay down with which to reason are immaterial, provided they are convenient for the objects we have in view. One feature must not be forgotten. We can never deduce the existence of phenomena through mathematical processes which were not implicitly contained in the laws of nature expressed at the outset in symbols. One cannot take out of the mathematical mill any product which was not present in the raw

material fed into it. It is the purpose of the mill to work up the raw material and the better the machinery the more finished and more varied will be the product.

To write a connected story of mathematical development within the limits of a brief article on a consistent plan without making it a mere catalogue of names and results is a difficult, perhaps impossible, task and no attempt is made here to accomplish it. If we try to lay stress on the workers rather than on what they achieved, in one period we encounter schools which developed particular subjects, in another, outstanding figures with or without influence on contemporary development and in still another numerous investigators who contributed in varying degrees to the advances made in their age. On the other hand if the development of ideas be the basis, parallel developments followed independently by different schools sometimes occur, at another time general methods of treatment seem to pervade and again we may find some fundamental advance made, the effect of which is not felt for many years. Consequently the plan which seems to fit best any particular period has been adopted for that period. Another difficulty consists in assigning the relative values to either men or ideas, about which probably no two persons will agree. There is, however, one stumbling block which is peculiar to mathematics. The very names themselves of many important branches of pure mathematics convey no meaning to the majority of scientific readers who are not trained mathematicians and to whom alone this article is intended to appeal. An attempt at brief definition has sometimes been made, but it can at best only give a partial view of the subject even with concrete illustrations of its significations. In the general outline, the historical development has been followed, but the methods of carrying it forward have varied with different periods. When a choice of names mentioned has to be made, a rough guide has been furnished in the earlier periods by selecting those who have taken the step forward which has rendered the subject capable of expansion or application by others as judged in the light of present knowledge. In the nineteenth and present centuries to carry out this method has proved to be beyond the ability of the writer; consequently, in most cases personal mention has only been incidental and the names of many of those who have done great service are missing. Fortunately, the history of mathematics has received much attention in articles and separate volumes and to them the reader who is interested in obtaining fuller information is referred.

The earliest traces in the form of written records have come to us from the Babylonians, mainly in the form of clay tablets which appear

to have been made at least 2000 years B. C. They show a knowledge of numbers which indicates that their civilization must have been far removed from the low stages in which many native tribes exist at the present day. Simple counting with the fingers of the two hands can be considered as the first stage, but beyond ten some new system must be devised. It appears that the Babylonians had learned the method of position, that is, that the first figure to the right shall represent the units, the next figure the tens and so on. They had even constructed numbers with 60 as the base of the system instead of ten. They could write numbers exceeding a million, one tablet giving a table of squares, and they also used fractions. Their geometry, however, was only in an elementary stage. But in astronomy they seem to have passed beyond the first stage of observation and to have been able to classify the results, for they possessed a knowledge of the Saros or period of 18, $\frac{2}{3}$ years in which the eclipses of the sun and moon recur; this must have involved a long period of observation and record as well as the ability to classify the results and it may perhaps be regarded as the earliest recorded scientific deduction from observation.

Concurrently with this civilization was one of perhaps equal development in Egypt. The Ahmer Papyrus, which is usually dated some 2000 years B. C., shows that the Egyptians had not only already constructed an arithmetic but had started the solution of what we now call equations of the first degree with one unknown. There was no general method for solving the problems and no symbolism for the unknown, although symbols for addition, subtraction and equality occur. In geometry they were also somewhat in advance of the Babylonians, apparently on account of practical needs in land surveying. It is generally agreed that the pyramids show evidence of astronomical observation in their orientation with respect to the stars and they certainly show evidence of a knowledge of geometrical form. As these monuments appear to go back some 4000 years B. C., we have evidence of some considerable development much further back even than the times indicated by the Babylonian tablets or the Ahmer Papyrus.

But the first real evidence of the scientific spirit comes from Greece. While they probably inherited some of the accumulated knowledge of both Babylonia and Egypt, they transformed much of the raw material thus acquired into a finished product which has survived to our own times. Much the most remarkable of all that we inherit from them in science, is the change which they effected in the study of geometry. The love and knowledge of form which is so strikingly exhibited in their buildings and statuary was also applied to geometrical figures. Development of logic and a real desire to know the sources of all things, was applied to the same study. Thus Greek geometry was not

only a consideration of the properties of straight lines and circles, but much more, a development of those properties by processes of thought from axioms laid down as a basis. If the modern mathematician can see defects of logic in much that has been handed down, he still follows Greece in the method of thought by which he deduces one result from another. Those same methods, indeed, are used to criticize the defects of the early work and that he is able to do so is chiefly due to the extended range of ideas which the developments of all forms of science have been able to give him. Some of the great names of antiquity in what we now term the humanities, are also great names in the history of science; Plato and Aristotle played no small part in the early developments.

The Greek school appears to have started about the seventh century before Christ, and it lasted nearly a thousand years, coming to an end finally after a long struggle against the stifling effect of the Roman conquest. For the first three hundred years Greece, a free nation, or at least under the government of its own citizens, found leisure to devote to intellectual pursuits, as a host of great names testifies. The invasion and repulse of Asia in the fifth century B. C. seems to have quickened rather than retarded the development of literature, science and the arts, perhaps under the spur of the construction of a single nation with democratic ideals from numerous small states. The conquest of Greece by Alexander in 330 B. C. transferred the sceptre nominally to Egypt through the enlightened policy of the Ptolemys who founded and encouraged schools of learning in Alexandria. But those who taught and worked there were all under Greek inspiration, and most of the advances were made by those of Greek origin. Apparently the foreign soil and alien patronage, however, gave only a temporary lease of life, for a decline started soon after and was accentuated in the first century B. C. when the Romans conquered Egypt and gained command of the whole civilized world. They themselves contributed little. The Fall of the Roman Empire through the incursion of the northern tribes, the rise and spread of Mohammed's followers, who, it is true, brought in other sources of knowledge from the eastern world but contributed little themselves, the domination and repression of free thought by ecclesiastics, left no opportunity for schools of science to grow. But few names appear in this period and those who sought to study the problems of nature had little opportunity to extend the borders of knowledge.

The first signs of awakening appeared towards the end of the fifteenth century. Revolts against ecclesiastical authority in England by the reigning sovereign, and later in Germany by Luther, the great geographical enterprises which soon resulted in a knowledge of the principal land areas of the earth, the curbing of royal ambitions by

the defeat of the Spanish Armada, and the brief interlude of a commonwealth in England, had their counterpart in the invention of printing and the appearance of scholars advancing knowledge in almost all the civilized countries of the western world. The tremendous steps taken in the course of two centuries, culminating in Newton and Leibnitz, finally placed scientific investigation on a plane where it became largely indifferent to what was going on in the political world, and where it was able to pursue its own course, hampered it is true, by wars and revolutions, but never without great names which will live as long as the history of scientific development is remembered.

This rapid sketch may serve to assist in seeing how the progress of scientific thought has been related to the development of civilization. We are all apt to regard our own concerns as independent developments and too often the history of science is treated in the same way. But in looking at the subject over long periods of time, we should treat scientific development as one of the phenomena which will illustrate progress or decline. The attitude of mind which leads to a search for the secrets of nature is simply one manifestation of a common desire for progress and if so, we should see signs of this desire in many directions. The ultimate causes which underlie the changes which have taken place—which produce periods of activity and inactivity in a whole people or group of peoples—are unknown and will probably only be finally found in the laboratories of the biologist and the physicist. All we can do now is to correlate the facts as far as possible and record them for future use.

Let us return to the Greeks and examine a little more in detail their contributions. In doing so we are at once faced by the difficulty that most of our knowledge comes to us second hand and in the form of treatises which gathered together past achievements. But few names survive and it is not always easy to apportion the credit. It sometimes appears that a name represents a school rather than an individual. This is certainly true to some degree of Pythagoras, to whom is usually credited the theorem that the sum of the squares on the sides of a right-angled triangle is equal to that on the hypotenuse. He certainly formed schools, but these were conducted as secret societies, the members of which might not divulge the knowledge they attained. Their continued existence for a long period of years was probably much assisted by the custom or rule of attributing all discoveries made by the members to their founder, thus avoiding much heartburning and jealousy. Nevertheless Pythagoras seems to have been responsible for placing geometry on a scientific basis by investigating the theorems abstractly. Briefly stated, it may be said that up to the middle of the fifth century B. C., shortly after the battle of Salamis, the Greeks had discovered the chief properties of areas in a plane and most of the regular solids.

Their clumsy notation for numbers handicapped them in arithmetic, but they knew such properties as that the difference of the squares of two consecutive numbers is always an odd number, they defined progressions of different kinds, and they had some acquaintance with irrational numbers. This was perhaps the first school which devoted itself to investigation for its own sake without any special reference to possible applications to physical problems.

Then followed the golden age of Greece with Hippocrates of Chios, Euclid, Archimedes, Appolonius, Hipparchus, and a number of others less well known. Plato and Aristotle must also be included for their contributions to the forms of logical reasoning which should be adopted, and Plato in particular contributed in this respect to an extent which has lasted until our own times. Euclid, whose personality is decidedly nebulous, wrote a text book on the geometry of the line and circle by which most mathematicians for the next two thousand years were introduced to the subject: it is indeed only during the last few decades that it has been replaced by modern texts. Appolonius did much the same thing for the curves of the second degree—the ellipse, parabola and hyperbola. While we know little of Euclid's own contributions as a discoverer, it is fairly certain that Appolonius had not only mastered all that was previously known, but greatly extended that knowledge himself. But of all those who lived up to the time of Isaac Newton, there can be little doubt that Archimedes is the chief. He is recognized as the founder of mechanics, theoretical and practical. His work on the lever alone entitles him to fame: "give me a fulcrum and I will move the world." He initiated the sound study of hydrostatics, advanced geometry by discovering how to find the area of a sphere and that cut off from a parabola by a straight line. He also discussed spiral curves, finding many of their properties. In arithmetic he seems to have had methods for dealing with very great numbers similar to those we now use by the index of the power of ten which can represent the number. These brief statements are only symbolic of what he achieved. His activities were very extended, and he preferred the modern custom of writing essays, or memoirs as we now call them, rather than treatises or text books. The tradition runs that he was killed at the fall of Syracuse to the Romans because, absorbed in a geometrical diagram, he insulted the Roman soldier who was spoiling it.

Hipparchus was mainly an astronomer, and indeed one of the founders of the art, and it was in the course of his work that he initiated Trigonometry as an aid when angles had to be measured as well as lines.

During the following centuries up to the Fall of Rome in A. D. 476, there were no great advances, the principal name of the period

being that of Diophantus, whose main contribution was a study of indeterminate equations. Algebra seems to have been developing very slowly and naturally, first by abbreviation of the words of a statement, next by a typical word (heap) and later by a symbol to represent the unknown, and finally by the adoption of symbols for common operations, like those of addition, equality, and so on. Diophantus took a great step in this direction, and he may be regarded as the father of Algebra in the main stream of the development of mathematics, in the sense that he placed it on a basis which rendered it capable of development. However, his work received no recognition at the time and was not revived for more than a thousand years.

We must now say something about some of the tributaries which paralleled the main Grecian stream and connected with it through the Moorish invasion of Europe during the 7th and 8th centuries. During the time of their dominance the Romans, if they contributed little, at least allowed development to continue in the centers of civilization. The Arabs, however, following the example set by the theologians, who by this time were beginning to come into power, had little use for scientific works and investigation, and between them they destroyed completely the greatest library and museum of antiquity, that of Alexandria, so that most of the written learning of that and of previous ages was lost for all time. They did however form a valuable connection with the Hindu mathematics of their time and appeared not only to have brought it to Europe, but to have assimilated it themselves to some considerable extent.

The origin of Hindu mathematics seems very uncertain. Apparently some centuries before our era, they had developed arithmetic further than the Grecians of the same period, but we do not know to what extent the Persian conquest might have been responsible for introducing early Greek learning into India. It is however true, that from whatever source they started, they developed arithmetic and geometry from the mensuration point of view further than the Greeks who had considered geometry from an abstract standpoint. In illustration of this, it may be mentioned that they had found good approximations to $\sqrt{2}$ and to π . But their greatest contribution is our present number system, which came to Europe through the Arabs and hence got its name. From the seventh century it does not appear that they formed an independent school. Hindu mathematics seems to have been largely improved by the needs of astronomy. Their greatest exponents were Brahmagupta, who lived in the seventh century A. D., and Bhāskara, some five centuries later. Their work, again, is chiefly arithmetical. While the so-called arabic numbers were probably used by the former, the latter was the first who is known to have given a systematic treatment of the decimal system. Although the Arabs dispersed the western

schools, they set up schools of their own which developed mainly on algebraical lines. One name stands out prominently, that of Alkarismi—who may be regarded perhaps as the founder of modern algebra—the name of the subject itself comes from him. But he used no complete symbolism, and he and his successors developed it mainly from the arithmetical point of view. In this school the modern names of the trigonometrical functions appeared although little was done in the way of development.

Chinese science seems to have started about the same time as that of Greece, but to have had little or no connection with our western science until the sixteenth century. Whatever mathematics the Japanese had probably came from China. It would appear at first sight that the earliest developments arose independently in all civilized nations about the same time, speaking broadly, but our knowledge is so scanty that we can only say that the records nearly all date back to similar periods in each case. Whether there is anything significant in this fact must be left to conjecture.

With this brief sketch this era may be thought of as closed in so far as the development of science and particularly mathematical science is concerned. It witnesses a real beginning in the study of geometry and algebra, and to a much less extent, of physical principles. A system of logical reasoning was discovered which, in its main outlines, still forms the basis of all deduction at the present day. The properties of the simpler geometrical figures had been studied and committed to writing. The advance of arithmetic was hindered by a poor numerical notation, but the foundations were laid for future development by the introduction of the Hindu symbolism. Algebra had started to emerge from the rhetorical form of discussion into the more terse abbreviations which we now use. Astronomy never failed to have exponents, though many who doubtless desired to increase their knowledge were restrained by the superstition of the age and by ecclesiastical authority, which attempted to dictate the thoughts as well as the actions of men. In mechanics, Archimedes seems to have stood almost alone, largely, perhaps, because no scientific method in dealing with the fundamental problems of nature had yet become current amongst the learned men.

It is difficult to sum up in a phrase the reasons for the scientific hiatus which occurred during the following three or four centuries. The revival of learning which sprang up towards the close of the eighth century was almost barren of progress in mathematics. The schools established by Charlemagne, while teaching mathematics, developed interest in other directions. Reverence for authority appears to be the basis of nearly all the learning of the age and there is no more stifling attitude of mind for progressive evolution of ideas. We may attribute

this to a variety of causes any one of which may seem sufficient to explain it, but in reality, not one of them explains anything. It may perhaps be best likened to one of those pauses which nature seems to demand and in which breath is sought before taking the next forward step.

The first sign of activity found expression in the Crusades undertaken to free Jerusalem from Mohammedan control. Soon after this time, that is, at the beginning of the twelfth century, several of the modern universities were founded, partly as gatherings of students to learn and discuss, and partly as developments of schools which had been under the charge of the monasteries. The old Greek works on mathematics were revived and the learning transmitted by the Arabs began to be assimilated and taught. Systematic instruction and the writing of treatises for the transmission of knowledge were begun. These were mingled with courses on astrology, alchemy, and magic, perhaps, we may conjecture, because the learned man had little chance to earn his living except through supplying what was in common demand, and perhaps also because the ancient learning gave little enough scope for cultivation of the imagination.

A century later Roger Bacon appears on the scene. It is difficult to overestimate the man who, travelling over Europe and studying in Paris and doubtless absorbing the learning of his time, becoming a Doctor in Theology and probably a monk, could break away from his training and absolutely reject the ideas and methods of his age. One who could write that in order to learn the secrets of nature we must first observe, that in order to predict the future we must know the past, must certainly have had unusual clarity of vision. He taught too that mathematics was the basis of all science, but he clearly recognized that it could not replace experiment and knowledge of phenomena, but could be used to great advantage in deducing results when the phenomena had once been observed: the point of view is thoroughly modern. But his three great works failed to have much influence on his times and seem to have been forgotten for several hundred years. Like many of his predecessors and followers in science, he suffered for his opinions.

The real start of modern science opens, as has been mentioned earlier, in the middle of the fifteenth century, and from this time on there is no break in the sequence of scientific discovery. We have also less need to relate its progress to that of the social order. While there have been periods in which wars have seriously disturbed the ability of nations to produce and to foster learning, there has been no time in the period in which the whole of the civilized world was so far involved in struggles that intellectual progress came to a stop everywhere. Moreover, all attempts since 1450 to enslave the world, either physically or spiritually, have ended in failure. It is true that five hundred years is comparable with the periods of Greek, Roman, and Mohammedan

supremacy, but in each of these civilizations we see signs of decline in a far shorter period. At present there appears to be no indication that the crest of the wave of scientific progress has been reached. The world war just concluded, in its essence a struggle for liberty of thought and action, was far too short to submerge the knowledge of the existing generation and prevent it from being handed on to the next.

The period can be roughly divided into two parts, the two centuries until the time of Newton forming the first of these. It is chiefly characterised by a sustained effort to lay solid foundations for all those sciences in which mathematics is needed for development, and success very nearly complete was attained at the end of the seventeenth century. In Astronomy and mechanics, Copernicus placed the sun in the center of our solar system. Kepler, building on the observations of Tycho Brahe, gave the laws of the planetary motions. Galileo laid foundations for the study of mechanics and proved the rotation of the earth about its axis, besides making a host of astronomical discoveries with his telescope. In mathematics, the arabic numerals came into full use and were applied to the needs of daily life. The symbolism of algebra was completed in nearly the modern form used in elementary mathematics: much of the work done by Vieta, Cardan and Tartaglia involved the solution of equations of the third and fourth degrees. Trigonometry, needed by the astronomer, the map-maker, and the navigator, was developed so far that Rheticus calculated a table of natural sines to every 10 seconds of arc and to 15 places of decimals. Logarithms were invented by Napier, and seem to have been adopted universally almost immediately. Descartes invented analytical geometry and thus gave to the investigators of space-forms a new weapon of immense power, while Desargues laid the foundation of projective geometry. Fermat, an amateur and perhaps as able as any of his contemporaries, laid down many of the laws of numbers and founded the calculus of probabilities.

The brevity of this summary is not a measure of the achievements of these two centuries- For this we must look to those which followed. While judged by modern standards, the actual progress seems small, it was far beyond that which had been achieved in all the previous ages. The only earlier contribution which has stood the test of time is perhaps the geometry of the Greeks, for the work of Archimedes, especially in mechanics, seems to have remained almost unknown up to the time of Galileo. The period showed not only a sound laying of foundations, but in its development of ideas, often only dimly expressed, showed that the germination of the seeds of future knowledge had already begun. Progress was frequently hampered by an unfortunate form of rivalry in which a discovery was kept back so that its possessor could propose problems to confute the claims for knowledge of his

contemporaries. On the other hand, the circulation of knowledge was greatly increased by the possibility of printing old and new work. Books became regular articles of merchandise and could even be picked up in out of the way places.

The era of Newton is so important in the history of both pure and applied mathematics that no excuse is necessary for dwelling on what was achieved. If an attempt be made to characterize its results in a single sentence, it may be perhaps best emphasized as the epoch of the discovery of the fundamental laws of continuously varying magnitudes. Before this time such solutions of dynamical problems as had been obtained were isolated. Newton's formulation of the laws of motion and proof of the law of gravitation were found in his hands and in those of his successors sufficient to deal with all the problems of physics which were then and later under consideration. Little progress, however, could have been made without the necessary complement, the calculus, which permitted of the study of varying quantities by symbolic methods. Rates of change, when uniform, presented little difficulty; the real problem was to deal with them when they were variable, as, for example, in the motion of a pendulum or the vibrations of a string. To a limited degree, the geometry of the straight line and the circle can deal with these questions as Newton showed in the classic translation into geometry of his results for the motions of the moon. But his manuscripts indicate that he failed beyond a certain point to give geometrical proofs of other results obtained by means of the calculus.

But as I have emphasized before we must not only look to the applications, the chief question in Newton's time, but also point out that varying magnitudes have been studied for their own properties so extensively that they form the larger part of mathematical developments up to the present day. If we except the science of discrete numbers, it is only in comparatively modern times that discontinuous magnitudes have received any extended study and even these have been advanced in many cases through developments of the calculus.

Isaac Newton seems to have been one of those very rare cases of genius breaking out without any very obvious stimulus from a particular teacher or school. His first introduction to mathematics was accidental: a book in astrology picked up at a fair in 1661 containing geometry and trigonometry induced him to study Euclid and then to continue by reading such text books as were available. His discovery of the calculus or "fluxions" as he called it, was made within three years and a half, the binominal theorem was formulated about the same time, and a year later he began his first attempts to prove the gravitation law. He was elected Professor of Mathematics in 1669, eight years

after his entry into Cambridge as an undergraduate. At this time his chief subjects for lectures and investigation were optics and algebra, the former of which involved him in much controversy. In fact all through his active period of work in mathematics, he seems to have suffered from the difficulties of having his great advances understood and accepted, although there never seems to have been much question amongst his contemporaries as to his wonderful powers. In 1679 he discovered the law of areas and showed that a conic would be described by a particle moving round a center of force under an attraction which varied inversely as the square of the distance. But it was not until 1684 that he began to work seriously at gravitation problems, his first step being to show that a uniform sphere exerts the same attraction as a particle of the same mass placed at its center. From this time on progress was rapid. Within two years the manuscript of the *Principia* was finished and the following year printed.

The *Principia*, like most of the works of the time consists partly of results previously known, but by far the larger part of it is Newton's own work. It begins with definitions, the formulation of the three laws of motion and the principal properties which can be deduced from them, with some examples, forming an introduction to the first book which contains his main work on gravitation. The second book is chiefly devoted to hydrodynamics and motion in a resisting medium and the third to various applications of the first book to bodies and motions in the solar system. As stated earlier, the proofs are cast into a geometrical form and freed from all traces of the method of fluxions which Newton had used to reach many of his results.

This great effort seems to have nearly exhausted his great powers for he produced little after its completion. In 1695 he accepted an appointment at the Mint and six years later resigned his chair at Cambridge. He was only forty-five when the *Principia* was published and he lived for thirty-eight years afterwards.

The half-century which followed the publication of the *Principia* seems to have been mainly occupied in understanding and digesting the advances made. On the continent, where Leibnitz' notation for the calculus was mainly adopted, a firm foundation was laid for the progress which began in the middle of the eighteenth century. In England the reverence for Newton was so great and separation from the continent so effective, that his methods dominated all the work for nearly a century and a half. This was perhaps mainly due to the translation of the *Principia* into geometry which ensured its early acceptance but must have fostered a distrust of all results which could not be proved in the same way. And further, Newton's notation for fluxions, which did not bring out their essential properties, had great influence in preventing advances as against that invented by Leibnitz. Nevertheless

there are certain names in the English school which have lived to the present day. Brook Taylor who was born in 1685 gave the fundamental series for the expansion of a function which is known by his name, followed a little later by Colin Maclaurin with the particular case which is named after him. The latter also determined the attraction of an ellipsoid and introduced the idea of equi-potential surfaces. De Moivre, of French ancestry but English birth and training, introduced the use of the imaginary into Trigonometry and thus prepared the way for the great development of the theory of functions of a complex variable which took place in the nineteenth century. Roger Cotes must also be mentioned if only for the high opinion Newton had of his abilities: he was only thirty-four years old when he died.

On the continent the Bernouilli brothers, friends and admirers of Leibnitz, were largely responsible for realising the power of the calculus and making it known. They applied it to many physical problems but perhaps their greatest influence came through their teaching abilities. Most of those in this period who achieved distinction were their pupils and several of their descendents were well known as mathematicians during the eighteenth century. But the most able men of this period were undoubtedly Clairaut and d'Alembert. The former produced the first theory of the motion of the moon developed from the differential equations of motion: in it he showed that the theoretical motion of its perigee, which in the *Principia* was obtained to only half the observed value, agreed with observation when we proceed to a higher approximation. There was an interesting development in this connection. Clairaut at first thought that it would be necessary to make an addition to the Newtonian law in order to produce agreement and it was only when he carried his work further that he saw such an addition to be unnecessary. A similar addition was examined by Newcomb and others as a possibility which might explain the deviation of the perihelion of Mercury from its observed value, and it is only within the last five years that such an addition has been deduced from the relativity theory by Einstein. Clairaut also obtained the well known formula for the variation of gravity due to the shape of the earth.

D'Alembert is best known by his work on dynamics in which he showed how the equations of motion of a rigid body could be written down: the principle is still known by his name. He also reached the well known wave-equation, a partial differential one of the second order, in several physical investigations and showed how a solution may be obtained. This was pioneer work but its further development was not carried forward to any extent by him.

The great period of continental activity which began in the middle of the eighteenth century and contained the names of Euler, Lagrange,

Laplace, and many others of whom a brief mention only can be made, was characterised also by social ferment which has profoundly changed the basis of society. The French revolution with all its far reaching consequences took place in the middle of the time of greatest mathematical activity, and the Napoleonic wars, carried into almost every country of Europe, served to stir up intercourse between the nations to a degree which must have had much effect on all forms of scientific activity. At the same time, the American revolution produced a new body of political thought which has had its development in the formation of a great and powerful nation. In England, comparatively free from invasion or social disturbance, little was produced of permanent value, the influence of Maclaurin being directed to the retention of published Newtonian methods. With the single exception of the Italian Lagrange, the great names of the period belong to France and Switzerland.

Those who have had the most far reaching influence, judged by modern standards, besides Euler, Lagrange, and Laplace, are Legendre, Fourier, Poisson, Monge, and Poncelet. I omit those whose chief labors were more closely associated with experimental work. The means for publication in this period were becoming more extensive and it is less easy to discover what each man owed to personal meetings with his contemporaries. The principal academies of Europe were starting or had started their volumes of transactions, extended treatises could be published and circulated, and the scientific men had new opportunities to meet and discuss, even to some extent with those of other countries, especially on the continent. The more enlightened courts sought the services of the ablest scientific men and, when the latter did not mingle too much in politics, on the whole treated them well. Fortunately, at least for the leaders, their teaching was usually confined to small bodies of earnest students and they appear to have been little hampered by demands on their time and energy for administrative duties. It is interesting to note that in a time of political disturbance which was perhaps comparable to that produced by the great war, the main foundations of modern mathematics, both pure and applied mathematics, were firmly laid.

Leonard Euler, born in 1707 at Basle in Switzerland and educated in mathematics by Bernouilli, was perhaps the most industrious of all his contemporaries and it is difficult to say whether his services to mathematics are to be judged best by his excellent treatises on analysis, including the calculus, or by his original memoirs on applied mathematics. In the former, he followed up all that was known at the time, recasting proofs and setting the whole in logical order. In the latter, he is best known for his formulation of the equations of motion of a rigid body, for a similar service in the equations of motion of a fluid

and for his work on the theory of the moon's motion. With respect to the last it may be said that every modern method of treatment can be found to have started with Euler. He continued his work to the end of his life in 1783 in spite of losing his sight some fourteen years earlier and his papers by a fire in 1777.

J. L. Lagrange was born at Turin in 1736 and was, like Newton, practically self-educated as far as his mathematical studies were concerned. It gives some insight into the comparatively small body of mathematical literature at the time he was seventeen years old and his own great ability, that an accident directed his taste for mathematics and that after two years work he was able to solve a problem in the calculus of variations which had been under discussion for half a century. At the age of twenty-five his published work showed that in ability he had no rival. Before his death, at the age of seventy-seven, he had left his influence on almost every department of pure and applied mathematics. His generalizations of the equations of mechanics have proved to be fundamental in all modern investigations and his applications to dynamical theory of the principle of virtual work and of the calculus of variations are now even more important than at any time in the past. The latter is applied not only to particles and rigid bodies, but also to fluids. To celestial mechanics he contributed several new methods in both the theoretical and practical sides of the subject in which such topics as the general problem of three bodies, stability of a planetary system, mechanical quadratures and interpolation are developed. In pure mathematics his lectures on the theory of analytic functions, afterwards expanded in treatises, form the basis on which later writers built. He also founded the science of differential equations by considering them as a whole rather than a treatment of such special equations as might arise in particular problems. And he contributed some important memoirs on the theory of numbers. His influence was undoubtedly much increased by a remarkable gift of exposition both in lecturing and writing: those who have read his *Mécanique Analytique* in which his most important dynamical contributions were placed will appreciate this fact. And this may be said independently of the simpler problem before him than has the modern mathematician with the enormous mass of past work which he has to consider and the selection which must necessarily be made.

P. S. Laplace, whose mathematical ability is unquestioned, was essentially an applied mathematician in that he devoted himself almost entirely to the solution of the problems of nature by mathematical methods. In general, he was not particular about mathematical proofs or logic, provided he could obtain results: in many respects he may be said to be the founder of the more modern schools of mathematical physicists. His most enduring work has proved to be in the theory of

attraction, especially gravitational, and its application to the solar system. He made potential a real and valuable instrument of analysis and discovery, working out many of its properties and applying it in various directions. The famous second order differential equation $\nabla^2\psi=0$, which is satisfied by every gravitational arrangement of matter, has been used as a substitute for the simpler expression of the Newtonian law of attraction and is especially interesting at the present time, since it may be regarded as the Newtonian analogue of the Einstein law. Laplace was the first to attempt a complete explanation of the motions of the bodies of the solar system or at least to formulate methods which could be applied for this purpose and his *Mécanique Céleste* remained the standard work of reference in this subject for a century. The theory of the development of the solar system from a primeval nebula which goes by his name and which was independently set forth somewhat earlier by Kant, has never been entirely rejected, although its supporters have often changed it almost beyond recognition while retaining his name in connection with their work. Its fundamental idea consists of the contraction of matter under gravitational attraction, but few now believe that the matter can produce planets and satellites by the throwing off of concentric rings as he supposed. What he did for celestial mechanics, he also achieved for the subject of probability, his *Théorie Analytique des Probabilités* being the classic treatise in which the whole is put on a sound basis and developed in various directions. It must be added that he gave many theorems and results in pure analysis but in most cases these were invented to solve physical problems.

Of the remaining names in this period, Legendre was essentially an analyst, his work being mainly in the theory of numbers, which few mathematicians of this time altogether left alone, in integral calculus and elliptic functions, his treatises on these subjects being still consulted. Monge and Poncelet may be regarded as the founders of modern descriptive and projective geometry respectively. Fourier in his *Théorie Analytique de la Chaleur* enunciated the theorem for the expansion of a function in a periodic form which has had such immense value in the discussion of all periodic phenomena, and has now a literature of its own. Poisson's work in attraction is on similar lines to that of Laplace, whose natural successor he seems to be.

It is convenient to view the progress made in the nineteenth and twentieth centuries from two points of view: one, the development of the three great branches of mathematics, geometry, analysis and their applications to other studies; the other, the development of new ideas which have applications in many branches of mathematics. While it may be said that the former is more particularly characteristic of the

first half of the nineteenth century and the latter of the succeeding period, it would give a false idea of the method of progress to regard these as anything more than general tendencies. But the difficulty (mentioned earlier,) of conveying an understanding of the advances made in pure mathematics, even by one much more familiar with them than the writer, occurs in an exaggerated form in attempting a chronicle of the work of the past century. The task is far easier in applied mathematics because most of us have some acquaintance with the problems, and the ideas to be conveyed are not so far away from our every day experience. Consequently, in the former, I can do little more than point to a sign post here and there, occasionally indicate the course of the road which has been followed, or describe by an analogy or an example a result which has been obtained.

The older geometry which consisted in the investigation of figures which could be generated under some simple descriptive definition like lines, circles, or conic sections, was greatly extended by Descartes' invention of analytical geometry. In the latter an algebraic statement of the properties gave rise to various classes of curves which could be ordered according to the forms of algebraic statement. Their properties could be investigated with much greater ease. The way was opened also for the consideration of the different kinds of curves or surfaces which possessed some definite general property; the properties common to a given class of curves or surfaces; the relations which may exist between theorems in analysis and geometry; and so on. When the methods of the calculus were added, the range of investigation was again widely extended through its facility for dealing with the properties of tangents and curvature. The new results obtained were undoubtedly instrumental in stimulating investigation from the more purely geometrical point of view. The names of Desargues, Monge, and Poncelet have been already mentioned as the creators of the subject of projective geometry; their work was continued and, during the third decade of the nineteenth century, developed into a separate branch by Moebius, Plücker, Steiner and a host of writers who have followed them. Simple illustrations of the idea involved, are that of map-making in which we represent portions of the spherical surface of the earth on a plane, or that of the shadow of a figure cast by a ray of light. These simple ideas have been generalised by considering the common properties of figures which are projected according to some given law and more generally by correspondences between two or more figures. Another development is that of transformations which leave properties unchanged. It will be seen at once that measurements of actual lengths of lines or metric properties, as they are called, are not those which would ordinarily be unchanged by projec-

tion, but this difficulty was overcome by Laguerre, Cayley and their successors.

Differential geometry is in its essence a study of the properties of geometrical figures by investigating the properties of small elements of those figures. Such properties as curvature of a single curve or surface, and those which depend on classes, such as the envelope of a systems of curves, the surfaces which cut systems of surfaces at right angles are the natural subjects of investigation under this head. In 1828 Gauss published a memoir which immensely extended the range of this subject, by introducing new ideas which have been applied to such topics as the deformation of surfaces under given conditions and more particularly to those properties which remained unchanged by deformation. The subject is closely allied to many problems in physics.

Many futile attempts to deduce the axioms of parallels from the other axioms laid down by Euclid led Lobatchewski and Bolyai to see what would happen if a geometry free from this axiom were constructed. The results showed that it could be made quite consistent in all its theorems, that some of the Euclidean theorems would still hold while others would be changed or generalised. Their chief successor was Riemann who showed that all previous geometries were special cases of a more general system. In our own time the subject has been developed in the direction of finding the properties which are possessed by the different geometries and also by investigation of the different sets of axioms which can be used as a basis for constructing different classes of geometries. In the applications to physics the most important has been perhaps the recognition that our own space is not necessarily Euclidean and that we can only find out its nature by examining properties which we are able to observe and measure.

The new developments have not prevented further research on the older lines. Geometry is still much used as a convenient language for the development and expression of analytical results. As seen below, the plane is the home of the complex variable, but in the theory of functions of this variable, it has become many-storied with ladders reaching from one story to another. Most of our physical problems, however, demand the use of three dimensional space and here the complex variable is not sufficient because with our ordinary algebraic rules, there are only two different kinds of numbers, the real and the imaginary, which are used to deal with two different directions in a plane. Hence the theory of vectors which deals with straight lines in any number of dimensions and particularly three has been evolved and is coming more and more into use on account of its brevity and compactness. It requires, however, a new notation to be learnt and a certain degree of familiarity with the operations which are possible.

The older theory of numbers in general dealt with integers and to a less extent with fractions, square roots, etc., that is, numbers which could be formed out of the integers by the ordinary operations of arithmetic. Gauss opened the way to a more extended idea of the meaning which might be attached to numbers by introducing those which are the solutions of an ordinary algebraic equation of any degree, whose coefficients are rational; such numbers are called algebraic. They naturally introduced complex numbers and have properties such as divisibility more extended than those which play a large part in our ordinary number system. This soon demanded a theory of congruences which, in their simplest form are numbers which, when divided by a given factor (called the modulus), always leave the same remainder (a residue). The theory of forms was another development which arose. Since Gauss' time the ideas have been greatly extended to many other kinds of numbers in which special classes have special properties, and these classes are the main subjects of investigation.

The extensive development of the theory of functions of a complex variable is perhaps the most significant achievement of the last hundred years. The imaginary was always arising in such questions as the solutions of quadratic equations and in the new branches. The next step was to give a geometric interpretation of a complex number by showing that it could represent in a single symbol the two co-ordinates of a point in a plane. A function of a complex variable was therefore a function which could take values over an area as against one of a real variable which could only take values along a line. The majority of functions become infinite for one or more values of the variable and these infinities play the major part in the development of the theory. To Cauchy belongs the honor of starting the work in the third decade of the nineteenth century, examining such questions as the possibility of developing such functions in series, their integrals, and the actual existence of functions of different kinds. Closely following him, Weierstrass and Riemann developed Cauchy's ideas, the former by basing his arguments on a special form—a series of powers of the variable—and the latter by using geometrical and even physical ideas for progress. Their successors have merged these different modes of development and have continued to investigate with success the representation of a function under given conditions, and the limitations of a given function. At the same time special functions, particularly those known as elliptic, were being developed by Abel and Jacobi, and the latter extended them in a very general manner. We have now many groups of such functions, some of which have become of sufficient importance to have whole treatises devoted to them.

The study of functions of real variables during the past half century has been largely due to the critical spirit which has compelled

mathematicians to examine thoroughly the foundations of the calculus. It consists of "all those finer and deeper questions relating to the number system, the study of the curve, surface and other geometrical notions, the peculiarities that functions present with reference to discontinuity, oscillation, differentiation and integration, as well as a very extensive class of investigations whose object is the greatest possible extension of the processes, concepts, and results of the calculus." (J. Pierpont, *Bull. Amer. Math. Soc.*, 1904, p. 147).

Ever since the invention of the calculus, the relations between two or more variables which contained also their derivatives and known as differential equations, have been continually brought before the eyes of the mathematician by the physicist, owing to the fact that the simplest symbolic statement of nearly all physical problems has been in the form of one or more differential equations. For finding the derivatives or integrals which arose in his work, definite rules were generally available even if they demanded much calculation. But he failed to find such rules for most of his differential equations, and in fact they do not exist. The pure mathematicians, led by Cauchy, took up the question from other points of view asking, in particular, the nature of the function which is defined by a differential equation. This is naturally an extension of the theory of functions and the methods of the latter opened the way. But the questions are so difficult that only a particular form, known as the linear, has made any considerable progress; this form, however, does embrace a large number of the functions whose properties had been examined. In our own generation the subject has been extended by the consideration of equations in which integrals also occur; these again are necessary in certain physical problems.

Most of the progress which has been made in applied mathematics will be treated in the article on Physics, but in addition to the remarks at the close of this article some few words may be said of those branches in the development of which mathematics plays the larger part. The chief of these is the dynamics of a system of particles and rigid bodies. W. R. Hamilton and C. G. J. Jacobi, in the second quarter of the century, put the equations of motion of all such systems into forms which not only permitted of remarkable generalisation, but indicated new methods of integration which opened out research into the general properties of such systems. The later work has been mainly developments and applications of these methods. The particular branch of this subject known as celestial mechanics has been continued on the practical side by extended theories of the motions of the planets and the moon and on the theoretical side by investigations into the general problem of three or more bodies. In the former numerous writers have continued with increasing accuracy the work of

the eighteenth century: for the latter new foundations were laid by Poincaré some thirty years ago.

Hydrodynamics has had less success in its applications. The prediction of the tides, chiefly from the work of G. H. Darwin, and the relative equilibrium of liquids in rotation by him and Poincaré have advanced in a satisfactory manner, but the knowledge of the motions of bodies in actual fluids is still in an elementary condition, especially when an attempt is made to apply it to under-sea and air craft. This, of course, refers, not to the experimental side, but to developments from the equations of motion. An interesting, but now almost neglected subject is that of vortex rings in a perfect fluid, the main features of which were given by Helmholtz and Lord Kelvin, giving rise to a hypothesis, now abandoned, that the fundamental atoms of matter consisted of such rings. The motions of our atmosphere have so far defied attempts at explanation on any general plan. The theory of sound, on the other hand, in the hands of Helmholtz and Lord Rayleigh, has been well developed. The theory of elasticity is almost entirely a creation of the present century and has found many applications.

Many of the ideas which are now fundamental in mathematics have had their origin in an attempt to advance some particular branch. Development has proceeded to a certain stage by means of known methods and then stops, owing perhaps to mathematical difficulties or to a failure of those methods to cast further light on it. A new method of attack has then been evolved, showing new roads by which it may be explored, after a time leading to openings which enable the investigator to continue on the earlier lines. These new methods have then been seen to be applicable to various other branches, thus forming connecting links and shedding new light. Indeed, one not infrequently meets with a statement that all mathematics can be based on some one of these ideas. This may be true, but progress demands that the subject be cross cut in many ways: a new country may be opened out by one great highway, but it is only well developed by several main roads in different directions with numerous connecting branches. I shall take up certain of these ideas and try to indicate briefly their bearing on various mathematical topics.

Some illustrations have already been given of the effect which a critical examination of the logical processes used in mathematics has produced. In geometry, the examination of Euclid's axioms has led to the discovery of ideas of space other than those which were current in earlier times. In analysis, algebras have been constructed in which some of the familiar rules have been dropped or changed. The way was thus opened to the examination of the foundations on which mathematics rests. Here the work gets close to a consideration of the mode

in which the human brain can think. But without going into this question it is possible to indicate the general method followed at the present time. At the outset a set of statements, usually called axioms or postulates (there is a difference of opinion as to the exact meaning to be attached to these words), is made. These statements may be redundant but must not be contradictory: a complete system is one which contains all the statements necessary for the object in view but which has nothing unnecessary or redundant so that no statement or combination of statements can result in another of the set. To connect and deduce, a system of reasoning is also required. From this it will be seen that mathematical science has no necessary relation to natural phenomena, but that it can be regarded as solely a product of the brain and that its results are simply consequences which may be deduced from ideas without external assistance. The last half century has seen great progress made in clarifying our ideas and in the introduction of rigorous methods of argument. It has now extended to the phenomena of nature, particularly in the direction of a reconsideration of our ideas of time and space and in an examination of our powers of observation, as will be illustrated below.

The idea of invariants has permeated every branch of pure and applied mathematics. In its elementary forms it is not difficult to understand. Natural processes are subject to change but we can nearly always find certain features of them which seem to remain the same in the conditions under which we observe them, as for instance, mass and energy. In geometry, the ratio of the circumference of a circle to its diameter is constant, the ratio of the sections of any system of straight lines cut by three parallel lines is the same, certain properties of a system of curves remain unchanged under given conditions of deformation, and so on. In analysis, one of the commonest modes of investigation is to find out what expressions remain unchanged when a specified change is made in the symbols. It has been said that all physical investigations are fundamentally a search for the invariants of nature. The various terms which occur in modern algebra, such as discriminant, Jacobian, covariant, are special forms of the same fundamental idea.

The idea of permutations and combinations which few of us fail to meet with in our every day experience, was chiefly developed in earlier times from the point of view of the number of arrangements which could be made of a set of objects under certain specified conditions. The modern theory of groups is the natural successor of this subject, but as has so often happened, the point of view and its development have changed. If we have a set of symbols and replace one by another according to a specified law, we can consider what changes will leave unchanged certain combinations of those symbols, as well as

the number of such changes which can be made. In doing this we naturally make a connection with the theory of invariants. Such a set of changes is called a group. But a more fruitful idea has been obtained by considering what combinations amongst the symbols, using a specified law of combination, will always produce another of the symbols, and will produce nothing else. As a simple and familiar example, take the series of numbers 0,1,2, . . . with the law of addition. If we add any two of these numbers we always get another of the same series and we never get any other kind of number. The whole of the series is called a group from this point of view. The even numbers form a sub-group under this definition but the odd numbers do not because the sum of two odd numbers is not an odd number. If we use the same series, omitting the zero, with the law of multiplication instead of that of addition, we again have a group, but now the odd numbers form a sub-group. Again we may consider the operation of turning a straight line through 60° in a plane about one end of the line. There are obviously six positions of the line and in whatever one of the six positions we start, a turn through 60° will always give one of the other positions. The whole set constitutes a group.

The idea of a group of substitutions enabled Galois and Abel, about the middle of the nineteenth century, to open up the way to treat algebraic equations of a degree higher than the fourth and in fact to show that the methods used to solve equations of the second, third and fourth degrees could not in general be applied to those of the fifth and higher degrees. The quintic had long been a puzzle to mathematicians, all attempts to give a general solution in terms of radicals having failed. Later on, Sophus Lie applied the idea of groups to the solution of differential equations and was able to indicate the nature of the solutions in certain general classes. Before his time the methods for finding them had been disconnected and apparently without any common property. Another form of the group theory has been applied with success to the investigation of curves and surfaces and it is not too much to say that the idea has been one of the most fruitful in producing progress. "When a problem has been exhibited in group phraseology, the possibility of a solution of a certain character or the exact nature of its inherent difficulties is exhibited by a study of the group of the problem."¹

One of the most useful efforts of the nineteenth century mathematicians has been in the direction of proving the possibility or impossibility of performing certain operations or of solving certain problems, that is, in the investigation of existence theorems, as they are often called. The squaring of the circle or the trisection of an angle are two of the oldest of them and later arose the obtaining of a finite numerical expression for the number e which is the base of the Napier-

ian system of logarithms and which arises in numerous mathematical and physical investigations. The failure of attempts to solve these problems finally led mathematicians to consider whether they could not be proved to be insoluble under the conditions laid down. Complete success has rewarded them. We now know that with the use of the ruler and compasses alone, it is not possible to find a square which shall be exactly equal in area to that of a given circle, nor given any angle is it possible to construct the lines which shall divide it into three equal parts. The number e too cannot be exactly expressed by fractions or square roots or any other such simple numerical representations, though it can be approximated to as closely as we wish by decimals or in other ways. The labor of useless effort on the part of the mathematician is thus avoided, though we shall still probably continue to hear of those who claim to have performed the impossible. In our time it is quite usual as a part of an investigation to find included in the construction of some new function or in a new representation of a known function, a proof of its existence; especially in those cases where the possibility may be called into question. In celestial mechanics an important part of Poincaré's work consisted in proofs of the existence or non-existence of different kinds of motion and of different kinds of integrals. Indeed we have a considerable class of literature which consists solely in demonstrating the existence of functions or curves with little indication of the methods by which they may be constructed. The stimulating value of such researches in suggesting problems is often forgotten by those who, with some justice, complain of their dullness.

It is strange in connection with existence theorems that some of the problems, most simple in statement, are still unproved. Long ago Fermat stated that there are no whole numbers which will satisfy the statement that the sum of the n^{th} powers of two whole numbers is equal to the n^{th} power of a third whole number, except when n is equal to 2. This impossibility has been proved for all values of n up to 100 and for a few beyond, but no general proof has yet been given that it is universally true. Again, there is no general method which will enable us to pick out the prime numbers, that is, those which are not divisible by any other number except unity. In geometry we have the famous four-color problem in which it is desired to prove that a map consisting of countries of any shape and arrangement can always be painted with four colors so that no two adjoining countries will receive the same color. In these and similar cases, no exceptions to the statements have been found and there exist no complete demonstrations of their possibility or impossibility. It is of course assumed that if they are true

¹L. E. Dickson, Paris International Congress, Vol. 2, p. 225.

a proof can be constructed without a change of our axioms concerning number or space.

It will be seen from the sketchy remarks of the last few paragraphs that at least one outstanding feature of pure mathematics during the last century has been its emancipation from the trammels imposed by any necessity for application to physical problems. It is, nevertheless, necessary to say a few words about these applications, although the major part of the story naturally comes under the history of physics. Under the general term "applied mathematics," are included at least three methods of study. In the first and simplest, we translate the physical problems into symbols and deduce the consequences we desire by mathematical methods. The work consists, therefore, of little more than an argument on lines laid down by the mathematician. In the second, a study of the formulae and relations which have arisen from physical problems is made, without any special desire to apply them to the phenomena: as indicated above, much of the pure mathematics arose in this way, even before it was recognized that such study was a quite legitimate intellectual exercise. In the third, the mathematical processes used by the applied mathematician are studied in order to find out their limitations, the extent of their validity, what extensions they will admit, how more general methods may be obtained, the best manner of treatment, and so on. This is not by any means an infertile source of progress, as may be illustrated by Poincaré's work on the divergent series which are used to calculate the places of the moon and planets.

The most fundamental change in the attitude of applied mathematicians has been in the recognition and working out of the consequences of simple fundamental principles or laws. Foremost amongst the latter is that known as the conservation of energy, brought into prominence in the middle of the nineteenth century by the labors of Helmholtz and Kelvin. It is now regarded as the chief invariant of the universe and has been applied to every branch of physics. Owing to the various forms which energy can take and to the fact that we are practically compelled, in applying mathematics to a physical problem, to deal only with some partial phase of it rather than with the whole, we cannot always assume that the principle holds in a particular problem. But in the majority of such cases the energy which is lost or changed from the particular form which we are considering is small so that this loss may be neglected or allowed for. When the loss is zero, the differential equations of the problem admit of an integral which expresses this fact. Newtonian mechanics lead to two forms of energy, kinetic (that due to motion) and potential (that due to position) and the development of the mathematics of all material systems has been mainly based on this separation.

The principle of Least Action, enunciated by Maupertius in 1744 but first put into correct mathematical form by W. R. Hamilton a century later, is essentially one which demands mathematical treatment. The Action of a system is a certain function depending on the velocities and relations of its parts which takes a minimum value whenever the system moves under natural laws. The process of discovery of this minimum leads to the differential equations of motion of the system and thus includes a complete statement of the problem. Since the initial form of the function is an integral, its mathematical treatment consists in finding the least value of this integral and thus becomes a problem in the calculus of variations to which considerable attention has been given by pure mathematicians, especially during the last two or three decades. While the physical consequences of the principle have been less developed than those of energy, there appears to be a growing feeling as to its fundamental importance and the aid of the mathematician in solving the problems which it raises, will become increasingly necessary.

The study of the properties of a system containing a large number of particles not fixed relatively to one another, now generally studied under the term, statistical mechanics, has penetrated into several branches. It is to be understood here that the question is not one of finding the separate motions of the various particles but to try and find out such properties of the system as can be deduced from averages. It is probable that as long as we cannot observe the motions of the separate particles, we should be able to deduce in this way most if not all the properties of the system that we are able to observe. Maxwell and Boltzmann founded the subject from this point of view, applying it to the kinetic theory of gases, while J. W. Gibbs was largely responsible for its application to thermodynamics. In astronomy the present century has seen it applied to the motions and positions of the stars, thus opening the way to a knowledge of the outlines of the construction of the stellar universe. Mathematically these questions are obviously very similar to those parts of probability which deal with errors of observation and thus form a continuation of the development of that subject.

The mathematics of continuous media has received very complete development during the century and, besides the earlier investigations into the motions of fluids and elastic solids, has been applied to the so-called luminiferous ether and finally by Maxwell to the whole electric field. In all this work the continuity of the medium is a fundamental axiom involving the hypothesis that no action can take place without its presence. Further, the Newtonian laws of motion were assumed as fundamental and time and length were regarded as unchangeable separate entities. The classic experiment of Michelson and

Morley which showed that the velocity of light was apparently independent of the velocity of the medium in which it travelled, and observations on the motions of certain particles with very high velocities, started a reconstruction of ideas. It was possible to explain the results on the assumption that the length of a body depended on its velocity. It was then that Einstein sought to generalize Newton's equations of motion by making them entirely relative, not only for uniform velocity, but also for accelerated motion. By adding the assumption that the laws of nature should refer to all such systems of reference and by making the velocity of light a fundamental constant of nature, he was finally able to generalize the whole subject. Matter appears simply as a form of energy. Gravitation can be exhibited as due to a warping of space without the introduction of force, but if this is so, the Newtonian law requires a minute correction. The motion of the perihelion of Mercury and the bending of a light ray as it passes near the sun have given remarkable confirmation of this theory. His work crosscuts several subjects which previously had an interest only for the pure mathematician, in particular the theory of extensible vectors (tensors) and the theory of invariants. His differential equations for the gravitational field should supply mathematicians with problems of great difficulty and interest for some time to come. Those for the electric field are unchanged. A further interesting product of this work on relativity lies in the question of what we can or cannot observe and in what may be deduced from observation without the assumption of hypotheses. This is, of course, fundamental in all experiments, but it has received little attention as an exact science. Given that we can only observe certain properties of a function, what limitations is it possible to make in the construction of the function?

Finally the quantum theory of Planck, according to which energy is not infinitely divisible but is always received or emitted in exact multiples of a fundamental unit, is bringing forward the necessity for a calculus allied to that of finite differences as against the differential and integral calculus which depends in general on continuity. It is even suggested that not only energy, but also space and time have ultimate parts which cannot be divided. At present the mechanics of this theory is in a very nebulous state but as a statement of the results of observation it has had very considerable success. The construction of the atom is now generally exhibited as a kind of minute solar system, but there is as yet no indication how such a system can only permit of the limited number of motions required by the quantum hypothesis. It may perhaps be due to fundamental instabilities for we know little of the ultimate stability of most of the motions in the problems of even three particles. In any case, the field of work has approached one of the oldest of mechanical problems and the reaction of celestial mechanics on that of the atom should prove stimulating to both.